

# Analysis of a Disease Transmission Model with two Groups of Infectives

**M.R. Razvan**

*Institute for Studies in Theoretical Physics and Mathematics*

*P.O.Box 11395-5746, Tehran, IRAN*

E-mail: razvan@karun.ipm.ac.ir

Fax: 009821-2290648

## **Abstract**

In this paper, we give a complete analysis of an SIS epidemiological model in a population of varying size with two dissimilar groups of infective individuals. It is mainly based on the discussion of the existence and stability of equilibria of the proportions system and the result is in terms of a threshold parameter which governs the stability of the disease free equilibrium.

**Keywords:** AIDS, core group, endemic proportions, epidemiological model, global stability, varying population.

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## **1 Introduction**

The social mixing structure of a population or a group of interacting populations play a crucial role in the dynamics of a disease transmission. (See [3] and references therein.) In almost all attempts to combine epidemiological data with mathematical modeling, there has been a recognition of the need to consider the structure of social interactions among the individuals in the populations. (See [11] and references therein.) Many authors have

considered the multigroup models in which heterogenous subpopulations may participate to the epidemic process with different parameters [2]. For SIS type models, a rather complete analysis of existence and global stability of a nontrivial epidemic state has been carried out by Lajmanovich and York [7]. In their work, the size of each subpopulation is assumed to be constant.

A famous example for these subpopulations is the core group, i.e. the highly sexually active subgroups [5]. It has become increasingly clear that the transmission within and among core subgroups is an important factor in the transmission of HIV/AIDS [6]. In order to consider the core group in an SIS epidemiological model, we divide the population into two subgroups each of them consists of susceptible and infective individuals. One of these subpopulations can be viewed as the core group. In this paper we consider another type of core group that is *post-infection core group*, i.e. individuals that become part of the core group after being infected. This hypothesis is plausible for a contagious and fatal disease like AIDS. From the psychological perspective, this group may be classified as violent.

In this paper we examine an SIS model of disease transmission in a population of varying size with two dissimilar groups of infective individuals. One of these groups can be viewed as the post-infection core group. We also assume that the birth rate of susceptibles may be more than that of infectives. This is similar to the demographic assumption in [8]. This paper is mainly based on the discussion of the existence and stability of equilibria of the proportions system. First of all, in the next section, we introduce the model and some concepts of ODE's related to the system. In Section 3, we present some basic results concerning the nonexistence of certain types of solutions. In Section 4, we give a complete global analysis of the proportions system which is reduced to a planar system. The result is in terms of a threshold parameter which governs the stability of the disease free equilibrium.

## 2 The Model

In order to derive our model, we divide the population into three groups: Susceptibles,  $S$ , and two groups of infectives,  $I_1$  and  $I_2$ . We set  $N = S + I_1 + I_2$  which is the total size of the population and we use the following parameters which are assumed to be positive unless otherwise specified:

$b$ : per capita birth rate of susceptibles,

$b_1$ : per capita birth rate of infectives which is assumed to be  $\leq b$ ,

$d$ : per capita disease free death rate,

$\varepsilon$ : excess per capita death rate of infectives,

$\lambda_1$ : effective per capita contact rate of  $I_1$ ,

$\lambda_2$ : effective per capita contact rate of  $I_2$ ,

$\gamma_1$ : per capita recovery rate of  $I_1$ ,

$\gamma_2$ : per capita recovery rate of  $I_2$ .

We also assume that the susceptible individuals which have been infected, enter to the group  $I_1$  and  $I_2$  of proportions  $p$  and  $q$  respectively, hence  $p + q = 1$ .

The above hypotheses leads to the following system of differential equations in  $\mathbb{R}_+^3$ , where “ $\prime$ ” denotes the derivatives with respect to  $t$ , the time,

$$\begin{cases} S' = b_1 N + (b_2 - d)S + \gamma_1 I_1 + \gamma_2 I_2 - \lambda_1 \frac{I_1 S}{N} - \lambda_2 \frac{I_2 S}{N}, & (2-1) \\ I'_1 = p \left( \lambda_1 \frac{I_1 S}{N} + \lambda_2 \frac{I_2 S}{N} \right) - (d + \varepsilon + \gamma_1) I_1, & (2-2) \\ I'_2 = q \left( \lambda_1 \frac{I_1 S}{N} + \lambda_2 \frac{I_2 S}{N} \right) - (d + \varepsilon + \gamma_2) I_2, & (2-3) \end{cases}$$

where  $b_2 = b - b_1$  and  $\frac{\lambda_j I_j}{N}$  is of the proportionate (or random) mixing type [9], [4]. By adding the above three equations, the total population equation is

$$N' = (b_1 - d)N + b_2 S - \varepsilon(I_1 + I_2)$$

Setting  $s = \frac{S}{N}$ ,  $i_1 = \frac{I_1}{N}$  and  $i_2 = \frac{I_2}{N}$ , we arrive at the following system of equations:

$$\begin{cases} s' = b_1(1-s) + b_2s(1-s) + \gamma_1 i_1 + \gamma_2 i_2 + (\varepsilon - \lambda_1)i_1s + (\varepsilon - \lambda_2)i_2s, & (2-1)' \\ i'_1 = ps(\lambda_1 i_1 + \lambda_2 i_2) + \varepsilon i_1(i_1 + i_2) - (b_1 + \varepsilon + \gamma_1)i_1 - b_2 si_1, & (2-2)' \\ i'_2 = qs(\lambda_1 i_1 + \lambda_2 i_2) + \varepsilon i_2(i_1 + i_2) - (b_1 + \varepsilon + \gamma_2)i_2 - b_2 si_2. & (2-3)' \end{cases}$$

In order to determine the asymptotic behaviour of the solutions of this system of equations, we need the following concepts of *ODE's* related to our system.

Given an autonomous system of ordinary differential equations in  $\mathbb{R}^n$ ,

$$\frac{dx}{dt} = f(x), \quad (2-5)$$

we will denote by  $x(t)$  the value of the solution of this system at time  $t$ , that is  $x$  initially. For  $V \subseteq \mathbb{R}^n$ ,  $J \subseteq \mathbb{R}$ , we let  $V.J = \{x(t) : x \in V, t \in J\}$ . The set  $V$  is called positively invariant if  $V.\mathbb{R}^+ = V$ . For  $Y \subseteq \mathbb{R}^n$  the  $\omega$ -limit (resp. the  $\alpha$ -limit) set of  $Y$  is defined to be the maximal invariant set in the closure of  $Y.[0, \infty)$  (resp.  $Y.(-\infty, 0]$ ). We say that  $\gamma(t)$  is an orbit running from  $x_0$  to  $x_1$  if  $\lim_{t \rightarrow -\infty} \gamma(t) = x_0$  and  $\lim_{t \rightarrow +\infty} \gamma(t) = x_1$ . These two points must be equilibria and such an orbit is called heteroclinic orbit. When  $x_1$  coincides with  $x_0$ , it is called a homoclinic orbit. A closed curve connecting several equilibria whose segments between successive equilibria are heteroclinic orbits is called a phase polygon. By a sink we mean an equilibrium at which all the eigenvalues of the linearized system have negative real parts. Such a point is called a source if all of these eigenvalues have positive real parts. If some of these eigenvalues have positive real parts and the others negative real parts, then the equilibrium is called a saddle point and it is called nondegenerate if all of these eigenvalues are nonzero.

### 3 Some Basic Results

We start our analysis with some basic results about the system  $(2-1)' - (2-3)'$ . If we set  $\Sigma = s + i_1 + i_2$ , then  $\Sigma' = (1 - \Sigma)(b_1 + b_2s - \varepsilon i_1 - \varepsilon i_2)$ . Therefore the plane  $\Sigma = 1$  is invariant. We consider the feasibility region

$$D = \{(s, i_1, i_2) : s + i_1 + i_2 = 1, s \geq 0, i_1 \geq 0, i_2 \geq 0\}$$

which is a triangle and on its sides we have:

$$\begin{cases} s = 0 \implies s' = b_1 + \gamma_1 i_1 + \gamma_2 i_2, \\ i_1 = 0 \implies i'_1 = p\lambda_2 s i_2, \\ i_2 = 0 \implies i'_2 = q\lambda_1 s i_1. \end{cases}$$

It follows that  $D$  is positively invariant and the disease free equilibrium  $(1, 0, 0)$  is the only rest point on  $\partial D$ , the boundary of  $D$ . Indeed our vector field points inward on  $\partial D - \{(1, 0, 0)\}$ . So every solution of the system  $(2-1)' - (2-3)'$  which starts in  $\partial D - \{(1, 0, 0)\}$ , immediately gets into  $\overset{\circ}{D}$ , the interior of  $D$ .

From now on, we examine the dynamics of this system in the feasibility region  $D$ . The following theorem is a modification of Theorem 4.1 in [1], concerning the nonexistence of certain types of solutions.

**Theorem 3.1.** Let  $f$  be a smooth vector field in  $\mathbb{R}^3$  and  $\gamma(t)$  be a closed piecewise smooth curve which is the boundary of an orientable smooth surface  $S \subset \mathbb{R}^3$ . Suppose  $g : U \rightarrow \mathbb{R}^3$  is defined and is smooth in a neighborhood  $U$  of  $S$ . Moreover it satisfies  $g(\gamma(t)) \cdot f(\gamma(t)) \geq 0$  and  $(\text{curl } g) \cdot n < 0$ , where  $n$  is the unit normal to  $S$ . Then  $\gamma$  is not a finite union of the orbits of the system (2-5).

In order to apply the above theorem, we define  $g = g_1 + g_2 + g_3$  by

$$\begin{aligned} g_1(i_1, i_2) &= \left[ 0, -\frac{f_3(i_1, i_2)}{i_1 i_2}, \frac{f_2(i_1, i_2)}{i_1 i_2} \right], \\ g_2(s, i_2) &= \left[ \frac{f_3(s, i_2)}{s i_2}, 0, -\frac{f_1(s, i_2)}{s i_2} \right], \\ g_3(s, i_1) &= \left[ -\frac{f_2(s, i_1)}{s i_1}, -\frac{f_1(s, i_1)}{s i_1}, 0 \right], \end{aligned}$$

where  $f_1, f_2$  and  $f_3$  deduced by  $\Sigma = 1$  on the right hand side of  $(2-1)', (2-2)'$  and  $(2-3)'$  respectively. Now after some computations we get

$$(\text{curl } g) \cdot (1, 1, 1) = - \left( \frac{p\lambda_2}{i_1^2} + \frac{q\lambda_1}{i_2^2} + \frac{b_1 + \gamma_1}{i_2 s^2} + \frac{b_1 + \gamma_2}{i_1 s^2} \right).$$

**Corollary 3.2.** The system  $(2-1)' - (2-3)'$  has no periodic orbits, homoclinic orbits or phase polygons in  $\overset{\circ}{D}$ .

**Proof.** We use Theorem 3.1. for  $f = (f_1, f_2, f_3)$ . Here we have  $g \cdot f = 0$  and  $(\text{curl } g) \cdot (1, 1, 1) < 0$  in  $\overset{\circ}{D}$ .  $\square$

**Lemma 3.3.** The  $\omega$ -limit set of each orbit of the system  $(2 - 1)' - (2 - 3)'$  with initial point in  $D$  is a rest point.

**Proof.** Suppose the contrary, then the  $\omega$ -limit set has a regular point in  $\overset{\circ}{D}$ . Let  $x$  be such a point and  $h$  be its first return map. For a point  $y$  near  $x$  on the transversal, let  $V$  be the region surrounded by the orbit  $\gamma$  from  $y$  to  $h(y)$  and the segment between them. This region is known as Bendixon sack. (See Figure 3.1.)

Now by Stokes' theorem

$$\int \int_V (\text{curl } g) \cdot (1, 1, 1) d\sigma = \int_{\gamma} g \cdot f dt + \int_0^1 g(ty + (1-t)h(y)) \cdot (y - h(y)) dt.$$

Since  $g \cdot f = 0$  and  $h(x) = x$ , the right hand side of the above equality tends to zero when  $y$  tends to  $x$ . But the left hand side tends to the integral over the region bounded by the  $\omega$ -limit set. This is a contradiction since  $(\text{curl } g) \cdot (1, 1, 1) < 0$  in  $\overset{\circ}{D}$ .  $\square$

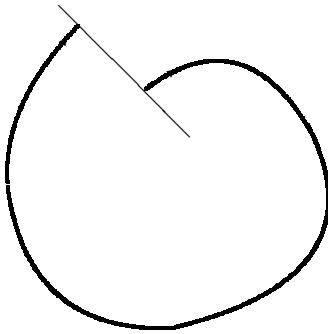


Fig. 3.1. The Bendixon sack.

**Remark 3.4.** When the  $\omega$ -limit set lies in  $\overset{\circ}{D}$  the above result is easily concluded by the generalized Poincaré-Bendixon theorem [10] and Corollary 4.2. Similarly if the  $\alpha$ -limit

set of an orbit of the system  $(2 - 1)' - (2 - 3)'$  lies in  $\overset{\circ}{D}$ , it must be a single point.

## 4 The Planar System

Using the equality  $s + i_1 + i_2 = 1$ , we see that our system is essentially two dimensional. Thus we can eliminate one of the variables, say  $s$ , to arrive at the following quadratic planar system

$$\begin{cases} i'_1 = (p\lambda_1 - b - \varepsilon - \gamma_1)i_1 + p\lambda_2 i_2 + (i_1 + i_2)((b_2 + \varepsilon - p\lambda_1)i_1 - p\lambda_2 i_2), \\ i'_2 = q\lambda_1 i_1 + (q\lambda_2 - b - \varepsilon - \gamma_2)i_2 + (i_1 + i_2)((b_2 + \varepsilon - q\lambda_2)i_2 - q\lambda_1 i_1). \end{cases} \quad (3-1)$$

The dynamics of the system  $(2 - 1)' - (2 - 3)'$  on  $D$  is equivalent to the dynamics of this planar system in the positively invariant region

$$D_1 = \{(i_1, i_2) | i_1 \geq 0, i_2 \geq 0, i_1 + i_2 \leq 1\}.$$

The matrix of the linearization of the system  $(3-1), (3-2)$  at the origin is:

$$C = \begin{bmatrix} p\lambda_1 - b - \varepsilon - \gamma_1 & p\lambda_2 \\ q\lambda_1 & q\lambda_2 - b - \varepsilon - \gamma_2 \end{bmatrix},$$

with  $\det C = (b + \varepsilon + \gamma_1)(b + \varepsilon + \gamma_2) - p\lambda_1(b + \varepsilon + \gamma_2) - q\lambda_1(b + \varepsilon + \gamma_1)$ . We set  $R_0 = \frac{p\lambda_1}{b+\varepsilon+\gamma_1} + \frac{q\lambda_2}{b+\varepsilon+\gamma_2}$ . Hence if  $R_0 < 1$ , then  $\det C > 0$  and  $\text{trace } C < 0$  and if  $R_0 > 1$  then  $\det C < 0$ . Thus we have proved the following lemma.

**Lemma 4.1.** Let  $R_0$  be the above threshold. Then the origin is a sink (resp. a saddle) for the system  $(3-1), (3-2)$  whenever  $R_0 < 1$  (resp.  $R_0 > 1$ ).

**Lemma 4.2.** The trace of the linearization of the system  $(3-1), (3-2)$  at a rest point in  $\overset{\circ}{D}_1$  is negative.

**Proof.** We compute the trace at a rest point in  $\overset{\circ}{D}_1$ .

$$\begin{aligned}\frac{\partial i'_1}{\partial i_1} &= p\lambda_1 - b - \varepsilon - \gamma_1 + (b_2 + \varepsilon - p\lambda_1 - p\lambda_2)i_2 + 2(b_2 + \varepsilon - p\lambda_1)i_1, \\ \frac{\partial i'_2}{\partial i_2} &= p\lambda_2 - b - \varepsilon - \gamma_2 + (b_2 + \varepsilon - q\lambda_2 - q\lambda_1)i_1 + 2(b_2 + \varepsilon - q\lambda_2)i_2.\end{aligned}$$

From  $i'_1 = 0$  and  $i'_2 = 0$ , we get

$$\begin{aligned}\frac{\partial i'_1}{\partial i_1} &= -p\lambda_2 \frac{i_2}{i_1} + p\lambda_2 \frac{i^2}{i_1} + (b_2 + \varepsilon - p\lambda_1)i_1 = -p\lambda_2 \frac{i_2}{i_1}(1 - i_2) + (b_2 + \varepsilon - p\lambda_1)i_1, \\ \frac{\partial i'_2}{\partial i_2} &= -q\lambda_1 \frac{i_1}{i_2} + q\lambda_1 \frac{i^2}{i_2} + (b_2 + \varepsilon - q\lambda_2)i_2 = -q\lambda_1 \frac{i_1}{i_2}(1 - i_1) + (b_2 + \varepsilon - q\lambda_2)i_2.\end{aligned}$$

Using the equality  $s + i_1 + i_2 = 1$ , we obtain

$$\frac{\partial i'_1}{\partial i_1} + \frac{\partial i'_2}{\partial i_2} = (b_2 + \varepsilon - \lambda_1)i_1 + (b_2 + \varepsilon - \lambda_2)i_2 - p\lambda_2 \frac{i_2 s}{i_1} - q\lambda_1 \frac{i_1 s}{i_2}.$$

Now from  $(2 - 1)'$  we have

$$s' = b_1(i_1 + i_2) + b_2 s(i_1 + i_2) + \gamma_1 i_1 + \gamma_2 i_2 + (\varepsilon - \lambda_1)i_1 s + (\varepsilon - \lambda_2)i_2 s = 0.$$

Thus  $(b_2 + \varepsilon - \lambda_1)i_1 s + (b_2 + \varepsilon - \lambda_2)i_2 s < 0$  and it follows that  $\frac{\partial i'_1}{\partial i_1} + \frac{\partial i'_2}{\partial i_2} < 0$ .  $\square$

The following two corollaries are immediate results of the above lemma.

**Corollary 4.3.** The system  $(3 - 1), (3 - 2)$  has no source in  $\overset{\circ}{D}_1$ .

**Corollary 4.4.** Every nondegenerate rest point of the system  $(3 - 1), (3 - 2)$  in  $\overset{\circ}{D}_1$  is hyperbolic.

**Remark 4.5.** A nondegenerate rest point of the system  $(3 - 1), (3 - 2)$  is obtained by a transversal intersection of the two conic sections  $i'_1 = 0$  and  $i'_2 = 0$ .

**Proposition 4.6.** There is at most one rest point in  $\overset{\circ}{D}_1$  for the system  $(3 - 1), (3 - 2)$ . Moreover such a rest point is always hyperbolic.

**Proof:** From the equilibrium conditions  $i'_1 = i'_2 = 0$ , we get the following equation which is homogeneous with respect to  $i_1$  and  $i_2$  of second order.

$$\begin{aligned} ((p\lambda_1 - b - \varepsilon - \gamma_1)i_1 + p\lambda_2 i_2)((b_2 + \varepsilon - q\lambda_2)i_2 - q\lambda_1 i_1) = \\ (q\lambda_1 i_1 + (q\lambda_2 - b - \varepsilon - \gamma_2)i_2)((b_2 + \varepsilon - p\lambda_1)i_1 - p\lambda_2 i_2). \end{aligned}$$

This equality can be written as

$$q\lambda_1(b_1 + \gamma_1)i_1^2 + (*)i_1i_2 - p\lambda_2(b_1 + \gamma_2)i_2^2 = 0$$

where  $(*)$  is a statement in terms of the involved parameters. The set of all roots of this quadratic equation consists of two lines through the origin in the  $(i_1, i_2)$  plane. One of these lines has negative slope and meets  $D_1$  only at the origin. Thus the other line contains all rest points of the system  $(3-1), (3-2)$  in  $D_1$ . Since each line contains at most two rest points of a quadratic planar system and this line contains the origin, it follows that  $\overset{\circ}{D}_1$  contains at most one rest point. This rest point is obtained by a transversal intersection of this line and each of the conic sections  $i'_1 = 0$  or  $i'_2 = 0$ . It is easy to see that at this rest point, these two conic sections must intersect transversally. Now by Remark 4.5, this rest point is nondegenerate and by Corollary 4.4, it must be hyperbolic.

□

**Remark 4.7.** We have indeed shown that all rest points of the system  $(3-1), (3-2)$  which are not more than three points, are nondegenerate, except the origin in the case  $R_0 = 1$ .

Now we are ready to prove our main result about the dynamics of the system  $(2-1)' - (2-3)'$  in  $D$ .

**Theorem 4.8.** (i) If  $R_0 \leq 1$ , then  $(1, 0, 0)$  is a global attractor in  $D$   
(ii) If  $R_0 > 1$ , then there exists a unique rest point (an endemic equilibrium) in  $\overset{\circ}{D}$  which attracts  $D - \{(1, 0, 0)\}$ .

**Proof.** When  $R_0 < 1$ , the origin is a sink for the planar system  $(3-1), (3-2)$ . If there exists another rest point in  $\overset{\circ}{D}_1$  for this system, it must be unique and hyperbolic. By

Corollary 4.3 it cannot be a source. If it is a sink, then we will have two sinks in  $D_1$ . The basins of attraction of these two points are open and by Lemma 3.3,  $D_1$  is the union of these two open subsets. This contradicts the connectedness of  $D_1$  and shows that it cannot be a sink. Now suppose that there is a saddle point in  $\overset{\circ}{D}_1$ . By Corollary 3.2, there is no homoclinic orbit in  $\overset{\circ}{D}_1$ . Hence the origin attracts the unstable manifold of the saddle point. Now the region bounded by the unstable manifold contains some part of the stable manifold of the saddle point. Thus the  $\alpha$ -limit set of this part of the stable manifold is a rest point in  $D_1$  by Remark 3.4. This is a contradiction since this rest point can be neither the origin nor the saddle point. Therefore the origin is the only rest point in  $D_1$  for the system (3 – 1), (3 – 2) and by Lemma 2.3, it is the  $\omega$ -limit set of all points of  $D_1$ .

The above fact is still valid for the limiting case,  $R_0 = 1$ . To see this, suppose that there exists another rest point in  $D_1$ , then it must be hyperbolic and belong to  $\overset{\circ}{D}_1$ . Thus it remains in  $\overset{\circ}{D}_1$  when the involved parameters are slightly changed to get  $R_0 < 1$  which contradicts the above result. This finishes the proof of (i).

Now suppose  $R_0 > 1$ . Then the origin is a saddle point for the planar system (3 – 1), (3 – 2). Thus by Lemma 2.3, there must be some rest point in  $\overset{\circ}{D}_1$ . Since such a rest point is unique and hyperbolic, it must be a sink and attract all points of  $D_1$  except the stable manifold of the origin. We claim that the stable manifold meets  $D_1$  only at the origin. To see this notice that some part of the unstable manifold of the origin must be outside of  $D_1$  for its right angle. Since  $D_1$  is positively invariant, the stable manifold does not intersect  $\overset{\circ}{D}_1$ . Moreover, the vector field points inward on  $\partial D_1 - \{(0, 0)\}$ . Thus the stable manifold of the origin does not intersect  $\partial D_1 - \{(0, 0)\}$  either. This shows that the origin cannot attract any point of  $D_1 - \{(0, 0)\}$ . Thus the unique sink attracts  $\overset{\circ}{D}_1 - \{(0, 0)\}$ . It means that there is a unique rest point in  $\overset{\circ}{D}$  for the system (2–1)'–(2–3)' which attracts  $D - \{(1, 0, 0)\}$ .  $\square$

**Remark 4.9.** In the above argument, in order to prove the global asymptotic stability of the endemic equilibrium (i.e. the unique rest point in  $\overset{\circ}{D}_1$ ), we showed that the stable manifold of the origin cannot intersect  $\overset{\circ}{D}_1$ . It is a special case of the following fact. Let  $X$  be a smooth vector field on a smooth manifold  $M$  and  $D \subset M$  is a positively invariant

region with a saddle point on  $\partial D$ . If the unstable manifold of this saddle point contains a point of  $(M - D)^\circ$ , then its stable manifold cannot intersect  $\overset{\circ}{D}$ . In order to prove it, one can follow our proof in the special case and observe that this is a direct consequence of the Hartman-Grobman theorem. However, it is obvious by the Inclination Lemma [10].

**Remark 4.10.** If we consider  $I_2$  as the post-infection core group, then the threshold  $R_0 = \frac{p\lambda_1}{b+\varepsilon+\gamma_1} + \frac{q\lambda_2}{b+\varepsilon+\gamma_2}$  clearly shows the effect of this group on the epidemics process. Although the probability  $q$  is a small number, the fact  $\lambda_2 \gg \lambda_1$  causes the term  $q\lambda_2$  to be significant.

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## References

- [1] Busenberg, S., Van den Driessche, P., Analysis of a disease transmission model in a population with varying size. *J. Math. Bio.* 28 (1990), 257-270.
- [2] Capasso, V., *Mathematical structure of the epidemic systems*. Lecture Notes in Biomathematics vol. 97, Springer-Verlag, Berlin, Heidelberg, 1993.
- [3] Castillo-Chavez, C., Cooke, K.L., Huang, W., Levin, S.A., On the role of long incubation periods in the dynamics of acquired immunodeficiency syndrome (AIDS). Part 2: Multiple group models. In: Castillo-Chavez, C. (ed.) *Mathematical and Statistical Approaches to AIDS Epidemiology*. (Lecture Notes in Biomathematics, Vol. 83) Springer-Verlag, Heidlberg, 1989.
- [4] Hethcote, H.W., van Ark, J.W., Epidemiological models for heterogenous populations: proportionate mixing, parameter estimation and immunization programs. *Math. Biosci.* 84 (1987), 85-118.
- [5] Hethcote, H.W., York, J.A., *Gonorrhea Transmission Dynamics and control*. Lecture Notes in Biomathematics vol. 56, Springer-Verlag, Berlin, Heidelberg, 1984.

- [6] Huang, W., Cooke, K.L., Castillo-Chavez, C., Stability and bifurcation for a multiple-group model for the dynamics of HIV/AIDS transmission. *SIAM J. Appl. Math.* 52 (1992), 835-854.
- [7] Lajmanovich, A., York, J.A., A deterministic model for gonorrhea in a nonhomogenous population. *Math. Biosci.* 28 (1976), 221-236.
- [8] May, R.M., Anderson, R.M., McLean, A.R., Possible demographic consequences of *HIV/AIDS* epidemics II. Assuming HIV infection does not necessarily lead to AIDS. In Castillo-Chavez, C., Levin, S.A., and Shoemaker, C.A. (eds.) *Mathematical Approaches to Problem in Resource Management and Epidemiology*. (Lecture Notes in Biomathematics vol. 81.) Springer-Verlag, Berlin, Heidelberg, 1989.
- [9] Nold, A., Heterogeneity in disease transmission. *Math. Biosci.* 52 (1980), 227-240.
- [10] Palis, J. and deMelo, W., *Geometric theory of dynamical systems, An introduction*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
- [11] Sattenpiel. L., The structure and social interactions and the spread of HIV. In: Castillo-Chavez, C.(ed.) *Mathematical and statistical Approaches to AIDS Epidemiology*. (Lecture Notes in Biomathematics, vol. 83.) Springer-Verlag, Heidelberg, 1989.